MEMORANDUM
RM-5368-PR
MAY 1967

NETWORKS, FRAMES, BLOCKING SYSTEMS

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UNITED STATES AIR FORCE PROJECT RAND

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This research is supported by the United States Air Force under Project RAND—Contract No. F44620-67-C-0045—monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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PREFACE

In this Memorandum, some basic problems concerning flow networks are surveyed and extended to two more general structures: frames of real vector spaces and blocking systems.

SUMMARY

This paper surveys some basic problems, theorems and constructions for flow networks, and shows how these can be extended to more general combinatorial structures.

One of the generalizations can be roughly described as that obtained by replacing the vertex-edge incidence matrix of an oriented network by an arbitrary real matrix. This leads to the notion of a frame of a subspace of Euclidean n-space, a concept very closely allied to that of a real matric matroid. Our treatment relates matroid theory and linear programming theory, and thus provides another viewpoint on linear programming, and in particular, on digraphoid-programming.

In the last part of the paper a very general combinatorial structure called a blocking system is given an axiomatic formulation. These systems have arisen in a variety of contexts, including multi-person game theory and abstract covering problems. It is shown that one of the network theorems surveyed in the first part of the paper extends to all blocking systems, and indeed characterizes such systems.

CONTENTS

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INTRODUCTION

In this paper we survey a few basic problems, theorems, and constructions concerning flow networks, and describe how some of these can be extended to more general structures.

The paper is divided into three parts.

Most of the material of Part I, which deals with networks, can be found in Ford and Fulkerson [8], or in earlier papers by the same authors. In the main, we limit the discussion in Part I to four network problems: maximum flow, minimum path, maximum capacity path, and the length-width inequality.

Part II extends this discussion to arbitrary real matrices by making use of what we call the frame of a subspace of Euclidean n-space, a notion very closely related to that of a real matric matroid. In particular, Part II can be specialized to a subclass of real matric matroids introduced and studied by Tutte [31], and called by him regular matroids. Regular matroids have been recently re-investigated by Minty [24], who has given another system of axioms for a dual pair of regular matroids. The resulting structure is called a digraphoid in [24], where it is shown that some of the main theorems of network-programming generalize to digraphoid-programming. Our treatment provides another viewpoint on digraphoid-programming, and indeed on

linear programming in general. It is shown in Part II that the main theorems of Part I have direct analogues for arbitrary real matrices. We want to emphasize, however, that the special network algorithms of Part I do not, so far as we know, have such analogues. Even for the case of digraphoid-programming, we know of nothing better computationally than the simplex method of Dantzig [3]. While the simplex method has proved to be a powerful tool, both theoretically and computationally, it is not yet known whether it is a good algorithm, in the technical sense stressed by Edmonds [6], whereas the network algorithms of Part I are good in this sense.

In Part III a very general combinatorial structure, which we call a blocking system, is given an axiomatic formulation. These systems have arisen previously in a variety of contexts, including multi-person game theory [29] and abstract covering problems [14, 21, 22]. They have recently been studied by Lehman [22], who has given conditions on a blocking system in order that a max-flow min-cut equality or a length-width inequality hold, and also by Edmonds and Fulkerson [7], who have shown that one of the network theorems of Part I extends to all blocking systems, and indeed characterizes such systems.

PART I. NETWORKS

1. MAXIMUM FLOW

Let G be a graph with edge set E and vertex set V. Both E and V are assumed finite. The two ends of an edge may be distinct vertices or the same vertex; in the latter case the edge is frequently called a loop. We also allow multiple edges joining the same pair of vertices, or multiple loops on the same vertex.

It will be convenient in this section to orient G by distinguishing one end of each edge as positive and the other as negative. For a loop these coincide. If $e \in E$ has positive end $u \in V$, negative end $v \in V$, we sometimes write $e \sim (u, v)$. For each edge $e \in E$ and vertex $v \in V$ we define an integer $a(v, e)$ as follows. If v and e are not incident, or if e is a loop, then $a(v, e) = 0$. Otherwise $a(v, e) = 1$ or -1 according as v is the positive or negative end of e. We call the resulting matrix the vertex-edge incidence matrix of G.

Suppose now that each edge $e \in E$ has associated with it a nonnegative real number $c(e)$, the capacity of e. Let s and t be two distinguished vertices of G. A (feasible) flow, of magnitude (or amount) α, from s to t in G is a real-valued function x with domain E that satisfies the linear equations and inequalities

$$(1.1) \qquad \sum_{e \in E} a(v, e)x(e) = \begin{cases} \alpha, & v = s, \\ -\alpha, & v = t, \\ 0, & v \neq s, t \end{cases}$$

(1.2) $$-\,c(e) \leq x(e) \leq c(e), \quad e \in E.$$

Thus $|x(e)|$ can be thought of as the magnitude of flow in edge e; if $x(e) > 0$, the direction of flow in e agrees with the orientation of e; if $x(e) < 0$, the direction of flow is against the orientation of e. The equations (1.1) stipulate that α units of flow leave s and enter t, flow being conserved at all other vertices. We call s the _source_, t the _sink_. The _maximum flow problem_ is that of constructing an x that satisfies (1.1), (1.2), and maximizes α.

We can get rid of the asymmetry in equations (1.1) by adding a special edge e' to G joining s and t, say $e' \sim (t, s)$, which returns α units of flow to s from t; we may take c(e') large. In other words, by distinguishing one edge e' of a graph, the maximum flow problem may be viewed as that of maximizing x(e') subject to (1.2) and the conservation equations

(1.1') $$\sum_{e \in E} a(v, e)x(e) = 0, \quad v \in V.$$

For the moment, we shall continue to work with (1.1) and (1.2), however.

We refer to the graph G with capacity function c and distinguished vertex pair s, t as a (_two-terminal_) _flow network_, or briefly, a _network_. In general, we use the word network in this paper to mean a graph together with one or more real-valued functions defined on its edges.

To state the fundamental theorem about maximum network flow, we require one other notion about graphs, that of a cut. A cut $K \subset E$ separating s and t in a graph G is a subset of edges that has some edge in common with each path joining s and t in G. We say that K blocks all such paths. (Here a path joining s and t is a sequence of distinct end-to-end edges that starts at s and ends at t. Edges may be traversed with or against their orientations in going from s to t along the path.) If all edges of K are deleted from G, the vertices s and t fall in separate components of the new graph. It is intuitively clear that α in (1.1) is bounded above by

$$c(K) = \sum_{e \in K} c(e), \tag{1.3}$$

the capacity of cut K. We can prove this from (1.1) and (1.2) by adding those equations of (1.1) corresponding to vertices in the s-component of the graph G' gotten from G by deleting edges of K. The result is

$$\alpha = \sum_{e \in K^+} x(e) - \sum_{e \in K^-} x(e) \leq c(K), \tag{1.4}$$

where K^+ (K^-) consists of those edges of K with positive (negative) end in the s-component of G' and negative (positive) end outside this component. In words, for an arbitrary flow from s to t of magnitude α and an arbitrary

cut separating s and t, the net flow across the cut is α, which is consequently bounded above by the cut capacity. Theorem 1.1 below asserts that equality holds in (1.4) for some flow and some cut, and hence the flow is a maximum flow, the cut a minimum cut [9].

Theorem 1.1. For any network the maximum amount of flow from source to sink is equal to the minimum capacity of all cuts separating source and sink.

Theorem 1.1, the max-flow min-cut theorem, is a combinatorial version, for the special case of the maximum flow problem, of the duality theorem for linear programs, and can be deduced from it [4]. Such a proof makes crucial use of the fact that the vertex-edge incidence matrix of an oriented graph G is totally unimodular, i.e., every square submatrix has determinant 0 or $\pm$ 1. A simpler proof of Theorem 1.1 is the second proof given by Ford and Fulkerson [10]. This proof also leads to an efficient algorithm for constructing a maximum flow.

Proof of Theorem 1.1: It suffices to establish the existence of a flow x and a cut K for which equality holds in (1.4). Let x be a maximum flow, of amount α, from s to t. Define a set $U \subset V$ recursively as follows:

(1.5a) $\qquad s \in U;$

(1.5b) if $u \in U$ and $e \sim (u, v)$ is an edge such that $x(e) < c(e)$, then $v \in U$; if $u \in U$ and $e \sim (v, u)$ is an edge such that $x(e) > -c(e)$, then $v \in U$.

We assert that $t \in \overline{U} = V - U$. For suppost not. It then follows from the recursive definition of U that there is a path P from s to t such that $x(e) < c(e)$ on edges $e \in P^+$ and $x(e) > -c(e)$ on edges $e \in P^-$. Here $P = P^+ \cup P^-$, where P^+ consists of those $e \in P$ whose orientations agree with the orientation of P from s to t. Let

(1.6) $$\epsilon = \min[\min_{e \in P^+} (c(e) - x(e)), \min_{e \in P^-} (c(e) + x(e))] > 0$$

and define

(1.7) $$x'(e) = \begin{cases} x(e), & e \notin P, \\ x(e) + \epsilon, & e \in P^+, \\ x(e) - \epsilon, & e \in P^-. \end{cases}$$

Then x' is a feasible flow from s to t of amount $\alpha + \epsilon$, contradicting the assumption that x was a maximum feasible flow. Hence $t \in \overline{U}$, as asserted. Let K be the set of edges joining U and $\overline{U}$, and write $K = K^+ \cup K^-$, where $K^+(K^-)$ consists of those edges of K with positive (negative) end in U. Then K is a cut separating s and t, and it follows from the definition of U that $x(e) = c(e)$ for $e \in K^+$,

$x(e) = - c(e)$ for $e \in K^-$. Hence equality holds in (1.4).

Notice that the proof shows that a flow x is maximum if and only if there is no x-augmenting path from s to t (i.e., a path P such that (1.7) yields a feasible flow x').

If we assume that the capacity function c is integral- (or rational-) valued, the proof provides a good algorithm for constructing a maximum flow. We can begin the computation with any integral-valued feasible flow from s to t, e.g., $x(e) = 0$ all $e \in E$. We then institute a search for a flow-augmenting path using the prescription of (1.5a) and (1.5b). A good way to apply this prescription is to fan out from s to all its neighboring vertices that can be put into U using (1.5b); then repeat the process by selecting one of these vertices, scanning it for all its neighbors not yet in U that can now be put into U, and so on. This way of searching for a flow-augmenting path is called the "labeling process" in [8], where it is described in terms of assigning labels to vertices as we put them in U; in terms of (1.5b), the label assigned to vertex v is u. (This simple process forms the basis for most of the network-programming algorithms described in [8].) If this search is successful in finding t, the flow increment ϵ of (1.6) is a positive integer, and hence x' of (1.7) is again an integral-valued flow. If unsuccessful, the present flow is a maximum flow, and a minimum cut has been located. Thus the algorithm terminates, and at termination

we have constructed an integral maximum flow and a minimum cut.

<u>Theorem 1.2</u>. <u>If the capacity function</u> c <u>is integral-valued, there is an integral maximum flow</u>.

Theorem 1.2 is important in combinatorial applications of network flows.

While we have taken the capacity constraints (1.2) to be symmetric about the origin, there is no real need for this assumption. The constraints (1.2) can be changed to

$$(1.2') \qquad b(e) \leq x(e) \leq c(e), \quad e \in E,$$

and handled in an analogous fashion provided they are feasible, that is, the constraint-set (1.1), (1.2') is nonempty. (Thus, for example, "one-way streets" can be incorporated in the model.) Even the feasibility question can be dealt with by an appropriate modification of the argument used in the proof of Theorem 1.1, or by applying a version of Theorem 1.1 to an enlarged network. For a detailed discussion of this and other extensions, e.g., capacities on vertices as well as edges, we refer to [8]. Here we shall simply state a typical feasibility theorem, the circulation theorem due to Hoffman [18].

<u>Theorem 1.3</u>. <u>Let</u> $b(e) \leq c(e)$ <u>for each edge</u> e <u>of a network</u> G <u>be given real numbers</u>. <u>The constraints</u> (1.1') <u>and</u> (1.2') <u>are feasible in</u> G <u>if and only if</u>, <u>for each subset</u> $U \subset V$, <u>we have</u>

$$\sum_{e \in K^+} c(e) - \sum_{e \in K^-} b(e) \geq 0,$$

<u>where</u> K^+ (K^-) <u>consists of those edges of</u> G <u>with positive</u> (<u>negative</u>) <u>end in</u> U <u>and negative</u> (<u>positive</u>) <u>end in</u> $V - U$.

Minty [23] has distilled from the above proof of the max-flow min-cut theorem and from other network algorithms of Ford and Fulkerson [10, 11] a theorem about graphs, which Berge and Ghouila-Houri [1] have called "Lemme des Arcs Colorés." We call it the painting theorem. To state it, we require some definitions. A <u>circuit</u> $C \subset E$ in graph G is a minimal closed path in G, that is, a set of edges which forms a closed path and is minimal with respect to this property. A <u>cocircuit</u> $D \subset E$ is a minimal cut, that is, a set of edges whose deletion increases the number of connected components of G and is minimal with respect to this property. (In terms of the $(0, \pm 1)$-vertex-edge incidence matrix of an orientation of G, a circuit corresponds to a minimal dependent set of columns of the matrix, where "dependent" means "linearly dependent over the reals." If G is unoriented, and the vertex-edge matrix

is taken to be a (0, 1) - matrix, then a circuit corresponds to a minimal dependent set of columns, where "dependent" means "linearly dependent over the integers mod 2.") A painting of G is a partition of the edges of G into three sets R, W, B, and the distinguishing of one edge of the set R. It may be viewed as painting the edges of G with three colors—red, white, blue—with one red edge being distinguished and painted dark red.

Theorem 1.4. *Given a painting of an oriented graph G, precisely one of the following alternatives holds:*

(i) *There is a circuit in G containing the dark red edge but no white edge, in which all red edges are similarly oriented.*

(ii) *There is a cocircuit in G containing the dark red edge but no blue edge, in which all red edges are similarly oriented.*

Proof: Let $e' \sim (t, s)$ be the dark red edge. If e' is a loop, then (i) holds and (ii) fails, by the minimality of a cocircuit. If $t \neq s$, define a subset $U \subset V$ recursively by the rules

(1.8a) $s \in U$;

(1.8b) if $u \in U$ and $e \sim (u, v)$ is red or blue, then $v \in U$; if $u \in U$ and $e \sim (v, u)$ is blue, then $v \in U$.

If $t \in U$, there is an elementary (minimal, simple) path from s to t of red and blue edges in which all red edges are oriented in the path direction. This path, together with edge e', provides the circuit of (i). Conversely, if (i) holds, then $t \in U$. If $t \notin U$, consider the set of edges joining U to $\bar{U} = V - U$. These edges are either white or red, and any red edge is oriented from $\bar{U}$ to U, as e' is. Delete these edges. The resulting graph has components U, $\bar{U}_1, \ldots, \bar{U}_k$ with $t \in \bar{U}_1$. The set of edges joining U and $\bar{U}_1$ is the cocircuit of (ii). Conversely, if (ii) holds, then t cannot be in U via (1.8b).

To apply the painting theorem to the maximum flow problem, first add the return-flow edge $e' \sim (t, s)$ to the network with c(e') large. Let x satisfy (1.1'), (1.2). Paint e' dark red. For other edges e: If $c(e) = 0$, paint e white; if $x(e) = c(e) > 0$, paint e red and reorient e; if $x(e) = -c(e) < 0$, paint e red; if $-c(e) < x(e) < c(e)$, paint e blue. Alternative (i) of the painting theorem then leads to a flow-augmenting path, whereas (ii) leads to a minimum cut. In this application the white edges play a pale role—they could have been deleted once and for all. But there are other network-programming problems for which labeling algorithms that have been described [10, 11, 12, 23] can be viewed in terms of edge paintings; the role played by white edges is less passive in some of these.

Before leaving the discussion of maximum network flow, we mention an alternative formulation of the problem. This formulation is in terms of the path-edge incidence matrix of an unoriented graph; it was used in the first proof of the max-flow min-cut theorem [9]. Let $\mathcal{P}$ be the collection of all paths from s to t in G. For each $P \in \mathcal{P}$ and $e \in E$ define an integer $p(P, e) = 1$ or 0 according as $e \in P$ or $e \notin P$. We call the resulting matrix the <u>path-edge</u> incidence matrix of G. Let y be a real-valued function with domain $\mathcal{P}$ that satisfies

(1.9) $$\sum_{P \in \mathcal{P}} y(P)p(P, e) \leq c(e), \quad e \in E,$$

(1.10) $$y(P) \geq 0, \quad P \in \mathcal{P}.$$

Thus y(P) can be thought of as the magnitude of flow in P, and (1.9) says that the total amount of flow in e cannot exceed its capacity. Subject to (1.9), (1.10), we wish to maximize

(1.11) $$\sum_{P \in \mathcal{P}} y(P).$$

This version of the problem might seem to be more restrictive, since if two paths P_1 and P_2 contain the same edge e in opposite directions, (1.9) insists that we add $y(P_1)$ and $y(P_2)$ instead of "cancelling flows in opposite directions."

The two formulations are equivalent, however.

If the capacity function c is integral valued, there is an integral-valued v satisfying (1.9), (1.10), and maximizing (1.11). An edge-form of Menger's theorem [20] can be deduced from this:

Theorem 1.5. *Let G be an unoriented graph with two distinguished vertices s and t. The maximum number of edge-disjoint paths joining s and t is equal to the minimum number of edges in a cut separating s and t.*

2. MINIMUM PATH

Let $\ell(e)$ be a real nonnegative number associated with edge e of an unoriented, connected graph G. We shall think of $\ell(e)$ as the *length* of edge e. The *length* of path P is

$$\ell(P) = \sum_{e \in P} \ell(e). \tag{2.1}$$

The second problem concerning two-terminal networks that we consider is the *minimum path problem*: to find a path joining s and t that has minimum length. There are several good methods known for doing this. We describe one below, but first we state and prove a theorem that is a path-cut dual of the max-flow min-cut theorem. Consider the maximum flow problem in terms of the path-edge incidence matrix. Suppose now that we form the *cut-edge incidence* matrix by defining $d(K, e) = 1$ or 0 according as $e \in K$ or $e \notin K$. Here K is a cut separating s and t. Let $\mathcal{K}$ denote the

class of such cuts. Analogously to (1.9), (1.10), let y be a real-valued function with domain $\mathcal{K}$ satisfying

(2.1) $$\sum_{k \in \mathcal{K}} y(K)d(K, e) \leq \ell(e), \qquad e \in E,$$

(2.2) $$y(K) \geq 0, \qquad K \in \mathcal{K} .$$

Again we wish to maximize

(2.3) $$\sum_{K \in \mathcal{K}} y(K)$$

subject to these constraints.

The maximum value of (2.3) cannot exceed the length of a minimum path from s to t, because a path from s to t has some edge in common with each $K \in \mathcal{K}$.

Theorem 2.1. *The maximum value of* (2.3) *subject to* (2.1) *and* (2.2) *is equal to the minimum path length from* s *to* t.

The purely combinatorial version of (2.1) – (2.3) in which $\ell(e) = 1$ all $e \in E$ and $y(K) = 0$ or 1 all $K \in \mathcal{K}$, asks for the maximum number of mutually disjoint cuts separating s and t. As was the case for the maximum flow problem, if ℓ is integral valued, there is an integral-valued y that solves the linear program (2.1) – (2.3). This will follow from the proof given below. Hence the maximum number of disjoint cuts separating s and t is equal to the minimum number of edges in a path joining s and t.

<u>Proof of Theorem</u> 2.1. Let $\pi(v)$ be the minimum path length from s to v, for all $v \in V$. Thus $\pi(v) \geq 0$ and $\pi(s) = 0$. Let $0 = \pi_0 < \pi_1 < \dots < \pi_n$ be the distinct values assumed by π. Partition V into n + 1 parts $V_0, V_1, \dots, V_n$, where

$$V_i = \{v \in V | \pi(v) = \pi_i\}.$$

Thus $s \in V_0$. Suppose $t \in V_k$. We then single out k cuts $K_1, K_2, \dots, K_k$ in $\mathcal{K}$ by letting K_j be the set of edges joining vertices of $\bigcup_{i=0}^{j-1} V_i$ and vertices of $V - \bigcup_{i=0}^{j-1} V_i$, $j = 1, 2, \dots, k$. Define $y(K_j) = \pi_j - \pi_{j-1}$, $j = 1, 2, \dots, k$, and $y(K) = 0$ for other cuts $K \in \mathcal{K}$. Then y solves (2.1) - (2.3). To prove this, it suffices to show that y satisfies (2.1), since clearly $y(K) \geq 0$ all $K \in \mathcal{K}$, and

$$\sum_{K \in \mathcal{K}} y(K) = \sum_{j=1}^{k} (\pi_j - \pi_{j-1}) = \pi_k - \pi_0 = \pi_k = \pi(t).$$

Thus consider an edge e joining a vertex u of V_i and a vertex v of V_j, where $i < j \leq k$, so that e belongs to each of the cuts $K_{i+1}, \dots, K_j$, but to no other cut having positive weight in y. Suppose that

$$y(K_{i+1}) + \dots + y(K_j) = \pi_j - \pi_i > \ell(e).$$

There is a path from s to u of length π_i; adjoining e to

this path yields a path from s to v of length $\pi_i + \ell(e) < \pi_i + (\pi_j - \pi_i) = \pi_j$, a contradiction. If j > k, a similar contradiction results. Hence y satisfies (2.1) and solves (2.1) - (2.3).

For the case of a planar two-terminal network (that is, the graph G together with the additional edge e' joining the terminals s and t is a planar graph), where one can construct a dual two-terminal network in which source-sink paths correspond to cuts separating s and t in the primal network, the duality between the maximum flow problem and the minimum path problem was noted in [9], and was exploited in developing a max-flow algorithm for such networks, the "top-most path" method of [9]. Theorem 2.1 for arbitrary two-terminal networks is due to Robacker [27]. From the point of view of Part II of this paper, Theorem 2.1 and the max-flow min-cut theorem are abstractly the same.

We return now to the problem of constructing a minimum path joining s and t. The procedure we sketch here is a special case of a more general algorithm for constructing minimum cost flows in networks [11]. It evaluates the minimum path length $\pi(v)$ from s to v for all $v \in V$, and hence provides a solution y to (2.1) - (2.3). We may suppose in the description that there are no loops or multiple edges in G. If edge e has ends u, v, we write the unordered pair (u, v) for e and $\ell(u, v)$ for $\ell(e)$.

To start out, take $\pi(s) = 0$. Next look at all edges (s, v) and find the minimum value of $\ell(s, v)$ for such edges. If v is a vertex yielding this minimum, set $\pi(v) = \ell(s, v)$.

The general step of the computation is as follows. Suppose that $\pi(u)$ has been defined for $u \in U \subset V$. Let $\bar{U} = V - U$ and compute

$$(2.4) \qquad \min_{u \in U, v \in \bar{U}} [\pi(u) + \ell(u, v)] = \delta .$$

If the minimum in (2.4) is achieved for an edge (u, v), set $\pi(v) = \delta$. Repeat the general step until $\pi(v)$ has been defined for all $v \in V$. The number $\pi(v)$ defined in this way is the minimum path length from s to v. A convenient way to do the calculation is to assign to vertex v the label $(u, \pi(v))$, where u is some vertex for which the minimum in (2.4) is achieved. A minimum path from s to v can then be found by backtracking from v to s as directed by first members of the labels.

At the conclusion of the computation, the numbers $\pi(v)$ satisfy the inequalities

$$(2.5) \qquad -\ell(u, v) \le \pi(v) - \pi(u) \le \ell(u, v)$$

for all edges (u, v) of G, and maximize $\pi(t) - \pi(s)$ subject to (2.5). If we interpret $\ell(u, v)$ as the cost of transporting a unit of some commodity over edge (u, v), the number $\pi(v)$ can be given the economic interpretation of a price placed on a unit of the commodity at location v. Inequalities (2.5) then say that no profit can be made by purchasing a

unit of the commodity at u and transporting it to v or vice versa. Subject to these restrictions, the price difference $\pi(t) - \pi(s)$ is to be maximized. Thus the maximum value of $\pi(t) - \pi(s)$ subject to (2.5) is equal to the minimum path cost from s to t. In another interpretation, Duffin has called this result the "max-potential equals min-work" theorem [5].

The assumption that edge lengths are nonnegative has been used in an essential way in this section. If edge lengths are allowed to be negative, and if we ask for a minimum length simple path joining two vertices, the problem is much harder. There are no known good algorithms for constructing such a path.

3. MAXIMUM CAPACITY PATH

Again we consider a two-terminal unoriented network G with source s, sink t, and capacity function c. This time we wish to find a path P from s to t that has the largest flow capacity of all such paths, i.e., we want to find a P that achieves

$$\max_{P \in \mathcal{P}} \min_{e \in P} c(e), \tag{3.1}$$

where $\mathcal{P}$ is the class of all paths joining s and t. We call this the <u>maximum capacity path problem</u>.

This bottleneck problem has been considered in [13, 19, 26]. It is related to the minimum path problem in the sense

that methods for solving the latter can be modified to solve it. But here we shall describe another easy way of solving the problem, one that extends to blocking systems (Part III). This method of solution might be termed the "threshold method." It leads to the following min-max theorem concerning paths and cuts [13].

Theorem 3.1. *Let G be a network with capacity function c and terminals s and t. Then*

$$(3.2)\qquad \max_{P\in\mathcal{P}}\ \min_{e\in P} c(e) = \min_{K\in\mathcal{K}}\ \max_{e\in K} c(e),$$

where $\mathcal{P}$ *is the class of all paths joining s and t and* $\mathcal{K}$ *is the class of all cuts separating s and t.*

Proof. If $P \in \mathcal{P}$ and $K \in \mathcal{K}$, then $P \cap K$ is nonempty. Let $e' \in P \cap K$. Then

$$\min_{e\in P} c(e) \le c(e') \le \max_{e\in K} c(e).$$

It follows that

$$(3.3)\qquad \max_{P\in\mathcal{P}}\ \min_{e\in P} c(e) \le \min_{K\in\mathcal{K}}\ \max_{e\in K} c(e).$$

To establish equality in (3.3), we can proceed as follows. Let $c_1 > c_2 > \ldots > c_n$ be the distinct values assumed by the capacity function, and let c_0 be large.

Let G_i be the network obtained from G by deleting all edges e satisfying $c(e) < c_i$, $i = 0, 1, \ldots, n$. Thus G_0 has no edges, and $G_n = G$. Suppose G_j is the first G_i that contains a path joining s and t. (We are tacitly assuming that $\mathcal{P}$ is nonempty, although an appropriate interpretation of (3.2) holds if this isn't so.) Since G_j has a path $P \in \mathcal{P}$ and G_{j-1} contains no path in $\mathcal{P}$, we have $\min_{e \in P} c(e) = c_j$. On the other hand, the edges deleted from G in forming G_{j-1} contain a cut $K \in \mathcal{K}$, whereas the edges deleted from G in forming G_j contain no cut in $\mathcal{K}$, and thus $\max_{e \in K} c(e) = c_j$. Consequently equality holds in (3.3).

Thus to solve the maximum capacity path problem, we lower the threshold for edge capacities until a path joining s and t is produced. There are good algorithms for recognizing when this happens.

Notice that no use is made of the fact that $c(e) \geq 0$. Indeed the solution depends only on the ordering of the edge numbers c(e), not on their magnitudes.

An appropriate version of the threshold method can be used to locate a flow-augmenting path that yields the largest flow increment (1.6). Thus one way to solve the maximum flow problem is to successively find maximum capacity flow-augmenting paths by a threshold method.

One can also show

$$(3.4)\qquad \min_{P\in\mathcal{P}} \max_{e\in P} c(e) = \max_{K\in\mathcal{K}} \min_{e\in K} c(e).$$

For an interpretation, think of G as a highway map with c(e) being the maximum elevation encountered in driving over edge e.

4. LENGTH-WIDTH INEQUALITY.

Duffin [5] has defined the notions of "extremal length" and "extremal width" for two-terminal networks having edge resistances and has shown that these are reciprocal quantities. From this relationship he deduced a certain inequality concerning paths and cuts for a two-terminal network in which each edge has associated with it two nonnegative numbers $\ell(e)$ and $w(e)$, the length and width of e. An earlier, purely combinatorial version of this inequality in which $\ell(e) = w(e) = 1$ is due to Moore and Shannon [25]. This version says that if λ is the least number of edges in a path joining s and t and ω is the least number of edges in a cut separating s and t, then $\lambda\omega$ is less than or equal to the number of edges in the graph. More generally, let

$$(4.1)\qquad \lambda = \min_{P\in\mathcal{P}} \ell(P) = \min_{P\in\mathcal{P}} \sum_{e\in P} \ell(e),$$

$$(4.2)\qquad \omega = \min_{K\in\mathcal{K}} w(K) = \min_{K\in\mathcal{K}} \sum_{e\in K} w(e),$$

where $\mathcal{P}$ is the class of all paths joining s and t, $\mathcal{K}$ is the class of all cuts separating s and t. The number λ

is called the <u>length</u> of G, ω the <u>width</u> of G, relative to s and t. The <u>length-width</u> <u>inequality</u> asserts that

$$(4.3) \qquad \lambda\omega \leq \sum_{e\in E} \ell(e)w(e).$$

A proof of (4.3) can be given using either the max-flow min-cut theorem or its path-cut dual. We use the former approach. Interpret w(e) as the flow-capacity of e. Then by the max-flow min-cut equality, there is a flow from s to t of magnitude ω. It follows that there is a function y defined on $\mathcal{P}$ satisfying (1.9), (1.10), and

$$\sum_{P\in\mathcal{P}} y(P) = \omega.$$

Thus

$$\lambda\omega = \lambda \sum_{P\in\mathcal{P}} y(P) \leq \sum_{P\in\mathcal{P}} \ell(P)y(P) = \sum_{P\in\mathcal{P}} \sum_{e\in P} \ell(e)y(P)$$

$$\leq \sum_{e\in E} \ell(e) \sum_{P\in\mathcal{P}} y(P)p(P, e) \leq \sum_{e\in E} \ell(e)w(e).$$

Although the length-width inequality appears weak, we shall point out in Part III that the existence of a length-width inequality for a blocking system implies the max-flow min-cut equality for the system.

PART II - FRAMES

Our aim in this part of the paper is to indicate how the theorems of Part I can be generalized to frames of subspaces of Euclidean n-space. (We shall define a frame later on. But it should be mentioned here that the word "frame" was used by Tutte in some of his early work on chain-groups and matroids in place of the word "matroid". We appropriate it, with his permission, for a more restrictive use.) The notion of a frame is closely related to that of a matric matroid. Indeed a frame can be viewed as the structure obtained just prior to the matroid in making the transition from matrix to its matroid.

Matroids were introduced by Whitney [35] as a generalization of dependence properties in graphs or in matrices. There is now an extensive and deep theory of matroids, mostly due to Tutte [30, 31, 32, 33, 34]. We require only the more elementary parts of this theory. (Certainly Tutte's <u>Introduction to the Theory of Matroids</u> [34] would suffice.)

The generalization from Part I to Part II can be described roughly as that obtained by replacing the vertex-edge incidence matrix of an oriented graph by an arbitrary real matrix. (More generally, we could consider matrices over any ordered field.) Thus we pass from the special network programs of Part I to general linear programs.

Associated with every linear program there is a dual program. Associated with every matroid there is a dual

matroid. Associated with every frame there is a dual frame. Frame duality provides a bridge between matroid duality and linear programming duality. The basic concept underlying duality in all three instances is orthogonality.

Although the material of this part of the paper was developed independently by the writer, we doubt that much of it is new. A recent paper by Rockafellar [28] contains a similar development, for example. Our attention has also been called to work of Camion [2], and to a forthcoming book on networks by Iri. Most of the notions and some of the results are either explicit or implicit in Tutte's work on matroids. We believe that our treatment of the generalized maximum flow problem and the resulting length-width inequality for real matrices may be new, however.

1. FRAMES OF REAL SUBSPACES

Let $\mathcal{R}$ be an arbitrary subspace of n-dimensional Euclidean space $\mathcal{R}^n$. For the correspondence with Part I, a vector $X = (x_1, x_2, \ldots, x_n)$ in $\mathcal{R}^n$ should be thought of as a real-valued function on a finite set of "edges" $E = \{e_1, e_2, \ldots, e_n\}$ that maps e_i into x_i, and $\mathcal{R}$ should be viewed as the row space of an m by n real matrix $A = (a_{ij})$, the "generalized vertex-edge incidence matrix".

Let $Y = (y_1, y_2, \ldots, y_n)$ be a vector of $\mathcal{R}$. The <u>support</u> $S(Y)$ of Y consists of those $e_i \in E$ such that $y_i \neq 0$. A vector $Y \in \mathcal{R}$ is called an <u>elementary</u> vector of $\mathcal{R}$ if it

is nonzero and if there is no nonzero vector $X \in \mathcal{R}$ such that $S(X)$ is a proper subset of $S(Y)$. Thus if X and Y are two elementary vectors of $\mathcal{R}$ having the same support, then X is a nonzero multiple of Y. Consequently we may associate with $\mathcal{R}$ a unique, finite set of lines, each line being determined by an elementary vector of $\mathcal{R}$. We call this collection of lines the <u>frame</u> $\mathcal{F} = \mathcal{F}(\mathcal{R})$ of $\mathcal{R}$, and sometimes refer to an elementary vector F of $\mathcal{R}$ as a frame—vector of $\mathcal{R}$.

Let X and Y be vectors of $\mathcal{R}$. The vector X <u>conforms to</u> Y if $x_i y_i > 0$ whenever $x_i \neq 0$. In particular, $S(X) \subset S(Y)$.

<u>Lemma 1.1</u>. <u>Let</u> Y <u>be a nonzero vector of</u> $\mathcal{R}$. <u>There exists an elementary vector</u> F <u>of</u> $\mathcal{R}$ <u>that conforms to</u> Y.

<u>Proof</u>: If not, select $Y = (y_1, y_2, \ldots, y_n) \in \mathcal{R}$ so that no elementary vector of $\mathcal{R}$ conforms to Y, and so that the number of elements in $S(Y)$ is as small as possible consistent with this condition. Let $X = (x_1, x_2, \ldots, x_n)$ be an elementary vector of $\mathcal{R}$ such that $S(X) \subset S(Y)$. Let $I \subset E$ denote the set of $e_i \in E$ such that y_i and x_i have opposite signs. Thus I is nonempty. Consider the vector $Z = Y + aX$, where

$$a = \min_{e_i \in I} \left(- \frac{y_i}{x_i}\right) > 0.$$

The vector Z conforms to Y and $S(Z)$ is properly included in $S(Y)$. By the selection of Y, there is an elementary

vector F conforming to Z. But then F conforms to Y. This contradiction establishes the lemma.

An important consequence of Lemma 1.1 is that any non-zero vector $Y \in \mathcal{R}$ can be written as a sum

$$(1.1) \qquad Y = F_1 + F_2 + \dots + F_k$$

of elementary vectors of $\mathcal{R}$, where each elementary vector F_i in (1.1) conforms to Y, and two elementary vectors F_i, F_j with $i \neq j$ lie on distinct frame-lines of $\mathcal{R}$. We call (1.1) a <u>conformal frame decomposition</u> of Y. In general, such a decomposition is far from unique, of course.

We return now to the matrix $A = (a_{ij})$ whose rows generate $\mathcal{R}$. A (<u>column</u>) <u>pivot</u> on an element $a_{k\ell} \neq 0$ of A is a sequence of elementary row operations on A that transforms A into a matrix $A' = (a'_{ij})$ in which $a'_{k\ell} = 1$, $a'_{i\ell} = 0$ for $i \neq k$. Starting with A, we can produce from it by a sequence of column pivots and deletions of zero rows a matrix R whose columns can be permuted to have the form

$$(1.2) \qquad (I, B).$$

If A has rank r, then R is r by n, the rows of R are a basis for $\mathcal{R}$, and R contains an r by r permutation sub-matrix whose columns correspond to some $S \subset E$. Following

Tutte, we refer to such a matrix R as a *standard representative matrix* of $\mathcal{R}$. Note that each row of R is an elementary vector of $\mathcal{R}$. The following theorem asserts that, conversely, any elementary vector of $\mathcal{R}$ can be obtained from A by a finite sequence of pivots.

Theorem 1.2. *Let* F *be an elementary vector of* $\mathcal{R}$. *Then there exists a standard representative matrix* R *of* $\mathcal{R}$ *having a multiple of* F *as one of its rows*.

Proof. Extend F to a basis $\mathcal{B}$ of $\mathcal{R}$, and write the resulting collection of vectors as a matrix having F as its first row, say. Pivot on a nonzero coordinate of F. Consider the second row of the transformed matrix. This row has a nonzero coordinate in one of the columns corresponding to zero coordinates of the first row, for otherwise either F would not be elementary or $\mathcal{B}$ would not be a basis. Pivot on such an element. Repetition of this process produces a standard representative matrix R of $\mathcal{R}$ having a multiple of F as its first row.

In particular, an elementary vector of $\mathcal{R}$ can have at most $n - r + 1$ nonzero coordinates.

Notice also that if $\mathcal{R}$ and $\mathcal{S}$ are subspaces having the same frame $\mathcal{F}(\mathcal{R}) = \mathcal{F}(\mathcal{S})$, then $\mathcal{R} = \mathcal{S}$.

2. MATROIDS

A matroid is a purely combinatorial structure defined on a finite set E. There are a number of equivalent axiom

systems for matroids. One in terms of "circuits" is as follows. Let $\mathcal{C}$ be a finite family of nonempty subsets of E. Members of $\mathcal{C}$ are the circuits of a matroid (E, $\mathcal{C}$) if the following axioms hold:

(2.1) No member of $\mathcal{C}$ is a proper subset of another.

(2.2) Let e_1 and e_2 be distinct members of E, and suppose C_1 and C_2 are members of $\mathcal{C}$ such that $e_1 \in C_1 \cap C_2$ and $e_2 \in C_1 - C_2$. Then there exists $C_3 \in \mathcal{C}$ such that $e_2 \in C_3 \subset (C_1 \cup C_2) - \{e_1\}$.

The motivation comes from graphs. Let E be the set of edges of an unoriented graph G. Then the collection $\mathcal{C}$ of (graph) circuits of G satisfies (2.1), (2.2), and thus (E, $\mathcal{C}$) is a matroid. Such a matroid is <u>graphic</u>. The collection $\mathcal{D}$ of cocircuits of G also satisfies (2.1), (2.2), and thus forms a matroid (E, $\mathcal{D}$). Such a matroid is <u>cographic</u>. For another important example, consider the row space $\mathcal{R}$ of the m by n matrix A. Take $E = \{e_1, e_2, \ldots, e_n\}$. Then the collection $\mathcal{C}$ of supports of frame-vectors of $\mathcal{R}$ satisfies (2.1), (2.2) and is consequently a matroid (E, $\mathcal{C}$). Such a matroid is called a <u>real matric matroid</u>.

Associated with every matroid (E, $\mathcal{C}$) there is a unique dual matroid (E, $\mathcal{C}^*$). A subset of E is a member of $\mathcal{C}^*$

if and only if the cardinality of its intersection with every element of $\mathcal{C}$ is not equal to 1, and it is minimal with respect to this property. The dual of the dual is the primal: $(E, \mathcal{C}^{**}) = (E, \mathcal{C})$. In case $(E, \mathcal{C})$ is a graphic matroid, the cographic matroid $(E, \mathcal{D})$ is the dual: $(E, \mathcal{D}) = (E, \mathcal{C}^*)$, $(E, \mathcal{D}^*) = (E, \mathcal{C})$. If $(E, \mathcal{C})$ is a real matric matroid arising from a subspace $\mathcal{R}$, the dual matroid is the real matric matroid obtained from the orthogonal complement $\mathcal{R}^*$ of $\mathcal{R}$. Thus if $\mathcal{F}$ is the frame of $\mathcal{R}$, we call the frame $\mathcal{F}^*$ of $\mathcal{R}^*$ the <u>dual</u> of $\mathcal{F}$. If $\mathcal{R}$ has standard representative matrix $R = (I_r, B)$, then a standard representative matrix for $\mathcal{R}^*$ is $R^* = (B^T, - I_{n-r})$. A frame-vector of $\mathcal{R}$ can be viewed as representing the coefficients of a minimal linear dependency among columns of R^*.

Let $A = (a_{ij})$ be the vertex-edge incidence matrix of an oriented graph G. It is well-known that the matrix A has the total unimodularity property: every square submatrix of A has determinant 0, 1, or -1. One can deduce from this that each elementary vector of the row space $\mathcal{R}$ of A is a multiple of a vector having coordinates 0, 1, or -1. Such a vector is called <u>primitive</u>. Conversely, if a subspace $\mathcal{R}$ has the property that each elementary vector of $\mathcal{R}$ is a multiple of a primitive vector, then $\mathcal{R}$ is the row space of some totally unimodular matrix $A = (a_{ij})$. In particular, $a_{ij} = 0$, 1, or -1. Such a space $\mathcal{R}$ is called

<u>regular</u> and the corresponding matroid is a <u>regular matroid</u>. Thus regular matroids are precisely those real matric matroids generated by totally unimodular matrices. The dual of a regular matroid is regular. A dual pair of regular matroids is called a "digraphoid" in [24].

(It should be remarked, though we make no use of it here, that Tutte has shown that a regular matroid is a binary matric matroid, that is, a matroid generated by a matrix over the field of two elements, and has characterized regular matroids as a subset of the binary matric matroids. This characterization, which is in terms of certain excluded matroid minors—a matroid minor is <u>not</u> the same thing as a matrix minor—is deeper than the one above, also due to Tutte, of regular matroids as a subset of real matric matroids. It can also be shown, as was pointed out to the writer by Edmonds, that a matroid is regular if and only if it is both a real matric matroid and a binary matric matroid. From this one can deduce that a $(0, \pm 1)$-matrix (I, B) is totally unimodular if and only if the binary rank of any subset S of its columns is equal to the real rank of S. This can also be proved directly. It is also possible to give a characterization of regular matroids among those real matric matroids generated by $(0, \pm 1)$-matrices in terms of a single excluded matroid minor: namely, exclude the self-dual matroid on a set of four elements, every triple of which is a circuit. The problem of characterizing

regular matroids among all real matric matroids in terms of excluded matroid minors appears to be open, as does the more fundamental problem of giving necessary and sufficient conditions in order that two real matrices generate the same matroid.)

The real matric matroid generated by the vertex—edge incidence matrix A of an oriented graph is a regular matroid. The nonzero coordinates of an elementary vector F of the row space $\mathcal{R}$ of A pick out a cocircuit in the graph, two edges being similarly oriented in this cocircuit if the corresponding coordinates of F have the same sign. Conversely, each cocircuit of the graph can be exhibited in this way as an elementary vector of $\mathcal{R}$. On the other hand, nonzero coordinates of an elementary vector of $\mathcal{R}^*$ pick out a circuit in the graph, two edges being similarly oriented in this circuit if the corresponding coordinates have the same sign, and each circuit of the graph can be exhibited in this way.

3. GENERALIZED FLOWS AND CUTS

Let $A = (a_{ij})$ be an m by n real matrix having row space $\mathcal{R}$. For each $e_j \in E = \{e_1, e_2, \ldots, e_n\}$, let c_j be a nonnegative real number, the capacity of e_j. In analogy with (1.1') and (1.2) of Part I, we define a (feasible) flow X on A to be a vector $X = (x_1, x_2, \ldots, x_n)$ that satisfies the linear homogeneous equations

$$(3.1) \qquad \sum_{j=1}^{n} a_{ij}x_j = 0,\ i = 1, 2, \ldots, m,$$

and inequalities

$$(3.2) \qquad -c_j \le x_j \le c_j,\ j = 1, 2, \ldots, n.$$

Thus $X \in \mathcal{R}^*$. Clearly feasible flows exist, e.g. $X = 0$.

The analogue of the maximum flow problem is to find a feasible flow X on A that maximizes some specified component of X, say x_1, where $c_1 = \infty$. We call such a flow a <u>maximum e_1-flow</u>.

Let $K = (-1, k_2, \ldots, k_n)$ be an elementary vector of $\mathcal{R}$. (Such elementary vectors exist unless the first column of A consists entirely of 0's—this corresponds to the graphic case in which e_1 is a loop.) We say that K is an <u>e_1-cut</u>. There are finitely many such. The <u>capacity</u> of an e_1-cut K is defined to be

$$(3.3) \qquad \sum_{\substack{j=2 \\ k_j>0}}^{n} k_jc_j - \sum_{\substack{j=2 \\ k_j<0}}^{n} k_jc_j = \sum_{j=2}^{n} |k_jc_j|.$$

If X is a feasible flow and K an e_1-cut, then, since $X \in \mathcal{R}^*$ and $K \in \mathcal{R}$, we have

$$\sum_{j=1}^{n} x_jk_j = 0,$$

and hence, by (3.2),

$$x_1 = \sum_{j=2}^{n} x_j k_j \leq \sum_{j=2}^{n} |k_j c_j|. \tag{3.4}$$

<u>Theorem 3.1</u>. <u>The maximum value of</u> x_1 <u>subject to</u> (3.1) <u>and</u> (3.2) <u>is equal to the minimum capacity of all</u> e_1<u>-cuts</u>.

<u>Proof</u>. It suffices to show that there is a flow and an e_1-cut for which equality holds in (3.4). A proof of this can be given using either the linear programming duality theorem [3, 16] or Dantzig's simplex method for solving linear programs [3]. We sketch the former approach. Let $X = (x_1, x_2, \ldots, x_n)$ be a maximum e_1-flow. The duality theorem for the linear program at hand then implies that there exists an m-vector $(\pi_1, \pi_2, \ldots, \pi_m)$ such that the following "optimality" properties hold:

$$1 + \sum_{i=1}^{m} \pi_i a_{i1} = 0, \tag{3.5}$$

and, for $j = 2, \ldots, n$,

$$\sum_{i=1}^{m} \pi_i a_{ij} > 0 \Rightarrow x_j = c_j, \tag{3.6}$$

$$\sum_{i=1}^{m} \pi_i a_{ij} < 0 \Rightarrow x_j = - c_j.$$

Let

$$Y = \left(\sum_{i=1}^{m} \pi_i a_{i1}, \ \ldots, \ \sum_{i=1}^{m} \pi_i a_{in} \right) = (-1, y_2, \ldots, y_n).$$

Thus $Y \in \mathcal{R}$. By Lemma 1.1, there exists an elementary vector $K = (-1, k_2, \ldots, k_n)$ of $\mathcal{R}$ that conforms to Y. The properties (3.6) then hold for K and imply that equality holds in (3.4). This proves Theorem 3.1.

The simplex method constructs a maximum e_1–flow and a minimum e_1–cut simultaneously. Indeed, the method proceeds by a sequence of pivots on A, and at termination yields a standard representative matrix R of $\mathcal{R}$, one of whose rows is an e_1–cut of minimum capacity.

If A is totally unimodular, then the coordinates of K in Theorem 3.1 are 0, 1, or –1, and we have a more purely combinatorial result: namely, the generalization of the max–flow min–cut theorem to regular matroids or digraphoids noted in [24]. Observe that the analogue of the integrity theorem, Theorem 1.2 of Part I, is valid for this case.

Just as for the case of flows in networks, the assumption of symmetric capacity constraints can easily be dispensed with in Theorem 3.1. The capacity constraints can be changed to $b_j \leq x_j \leq c_j$, and treated in a similar fashion, provided they are feasible. The capacity of an e_1–cut $K = (-1, k_2, \ldots, k_n)$ is then defined to be

(3.3') $$\sum_{\substack{j=2 \\ k_j>0}}^{n} k_j c_j + \sum_{\substack{j=2 \\ k_j<0}}^{n} k_j b_j .$$

The feasibility question is most conveniently disposed of by the following "generalized circulation theorem," the analogue of Theorem 1.3, Part I.

Theorem 3.2. Let $A = (a_{ij})$ *be an* m *by* n *real matrix, and let* $b_j \le c_j$, $j = 1, 2, \ldots, n$, *be given real numbers. The constraints*

(3.7) $$\sum_{j=1}^{n} a_{ij} x_j = 0, \quad i = 1, 2, \ldots, m,$$

(3.8) $$b_j \le x_j \le c_j, \quad j = 1, 2, \ldots, n,$$

are feasible if and only if, for each elementary vector $K = (k_1, k_2, \ldots, k_n)$ *in the row space* $\mathcal{R}$ *of* A, *we have*

(3.9) $$\sum_{k_j>0} k_j c_j + \sum_{k_j<0} k_j b_j \ge 0.$$

Notice that (3.9) is really a finite set of inequalities, since we need only choose from each frame-line of $\mathcal{R}$ one elementary vector and its negative in checking (3.9).

We turn next to the painting theorem for a real m by n matrix $A = (a_{ij})$. Here we paint the edges of $E = \{e_1, e_2, \ldots, e_n\}$, i.e. the columns of A, with three colors—red,

white, blue—with one red edge being distinguished and painted dark red. Two edges e_i, e_j are _similarly oriented_ in an elementary vector $X = (x_1, x_2, \ldots, x_n)$ of a subspace $\mathcal{R} \subset \mathcal{R}^n$ if $x_i x_j > 0$; X _contains_ e_i if $e_i \in S(X)$.

Theorem 3.3. Given a painting of $E = \{e_1, e_2, \ldots, e_n\}$ _and a real_ m _by_ n _matrix_ $A = (a_{ij})$ _having row space_ $\mathcal{R}$, _precisely one of the following alternatives holds_:

(i) _There is an elementary vector_ X _of_ $\mathcal{R}^*$ _containing the dark red edge but no white edge, in which all red edges are similarly oriented._

(ii) _There is an elementary vector_ Y _of_ $\mathcal{R}$ _containing the dark red edge but no blue edge, in which all red edges are similarly oriented._

Proof. Clearly both alternatives can't hold, since $\mathcal{R}$ and $\mathcal{R}^*$ are orthogonal.

Delete all white columns of A. Pivot on blue columns, one after another in any order, until no more such pivots are possible. (These operations correspond to the "deletions" and "contractions" of edges in graph or matroid theory.) Now delete all rows and columns of the resulting matrix that contain pivotal elements. Call the remaining matrix $\bar{A}$. Note that any blue columns of $\bar{A}$ consist entirely of 0's. ($\bar{A}$ generates a matroid minor of the matroid generated by A.) Let $\bar{\mathcal{R}}$ denote the row space of $\bar{A}$. It follows from standard theorems on linear inequalities that (just) one of a pair of complementary orthogonal subspaces contains a nonnegative

vector whose first coordinate, say, is positive. Thus either $\mathcal{R}$ or $\mathcal{R}^*$ (but not both) contains a nonnegative vector whose dark red coordinate is positive. Suppose $\bar{Z} \in \mathcal{R}^*$ is such a vector. Then $\bar{Z}$ can be extended to a vector $Z \in \mathcal{R}^*$ such that white coordinates of Z are all zero. In this case (i) holds, by Lemma 1.1. Suppose that $\bar{Z} \in \bar{\mathcal{R}}$ is a nonnegative vector whose dark red coordinate is positive. In this case $\bar{Z}$ can be extended to a vector $Z \in \mathcal{R}$ such that all blue coordinates of Z are zero. In this case (ii) holds, by Lemma 1.1.

If A is totally unimodular, the elementary vectors X and Y of Theorem 3.3 can be taken to be primitive, and Theorem 3.3 reduces to the painting theorem for digraphoids [24].

We return now to the version of Theorem 3.1 with capacity constraints $b_j \leq x_j \leq c_j$. How general is the class of linear programs encompassed by this theorem? The answer is not hard to see: it includes all linear programs. For, as is well known, any linear program can be put in the form

$$(3.10) \qquad \sum_{j=1}^{n} a_{ij}x_j = b_i, \; i = 1, 2, \ldots, m,$$

$$x_j \geq 0, \; j = 1, 2, \ldots, n,$$

$$\text{maximize} \sum_{j=1}^{n} c_j x_j .$$

Introducing new variables x_0 and x_{n+1}, we see that (3.10) is equivalent to

$$(3.11) \qquad \begin{aligned} &\sum_{j=1}^{n} a_{ij}x_j - b_i x_{n+1} = 0, \\ &x_0 - \sum_{j=1}^{n} c_j x_j = 0, \\ &-\infty \leq x_0 \leq \infty, \\ &0 \leq x_j \leq \infty,\ j = 1, 2, \ldots, n, \\ &1 \leq x_{n+1} \leq 1, \\ &\text{maximize } x_0. \end{aligned}$$

The program (3.11) is a maximum flow problem on a subspace of $\mathcal{R}^{n+2}$.

Still following the discussion of Part I, Section 1, let us look now at the general version of the path—edge formulation of the maximum flow problem. Is there an analogue of (1.9), (1.10), and (1.11) for an arbitrary real matrix? We shall see that there is. Consider the matrix whose rows consist of all elementary vectors of $\mathcal{R}^*$ of the form $(1, p_2, \ldots, p_n)$. Let (p_{kj}), $k = 1, 2, \ldots, s$, $j = 1, 2, \ldots, n$, denote this matrix and let $(|p_{kj}|)$ be the matrix obtained by taking absolute values of elements. We want

to show that the programs

$$(3.12) \qquad \sum_{k=1}^{s} y_k \cdot |p_{kj}| \le c_j, \quad j = 1, 2, \ldots, n,$$

$$y_k \ge 0,$$

$$\text{maximize} \sum_{k=1}^{s} y_k \cdot |p_{k1}| = \sum_{k=1}^{s} y_k,$$

and

$$(3.13) \qquad \sum_{j=1}^{n} a_{ij} x_j = 0, \; i = 1, 2, \ldots, m,$$

$$-c_j \le x_j \le c_j, \; j = 1, 2, \ldots, n,$$

$$\text{maximize } x_1,$$

are equivalent. Here we take $c_1 = \infty$. Given a feasible solution $Y = (y_1, y_2, \ldots, y_s)$ of (3.12), define $x_j = \sum_{k=1}^{s} y_k p_{kj}$. Then $-c_j \le x_j \le c_j$ and

$$\sum_{j=1}^{n} a_{ij} x_j = \sum_{k=1}^{s} \left(\sum_{j=1}^{n} a_{ij} p_{kj} \right) y_k = 0.$$

Conversely, given a feasible solution $X = (x_1, x_2, \ldots, x_n)$ of (3.13), we use the conformal frame decomposition (1.1) to write

(3.14) $X = y_1P_1 + \dots + y_\ell P_\ell + F_1 + \dots + F_h, \ y_k > 0,$

where $P_1, \dots, P_\ell$ are the first ℓ rows, say, of the matrix (p_{kj}), and each elementary vector in (3.14) conforms to X. Define $y_k = 0$ for the remaining rows of (p_{kj}). It follows that

$$\sum_{k=1}^{s} y_k \cdot |p_{kj}| \leq c_j .$$

Thus (3.12) and (3.13) are equivalent programs.

In particular, if A is totally unimodular, then $(|p_{kj}|)$ is a (0, 1)—incidence matrix, and an integral X in (3.14) yields an integral Y solving (3.12). Thus integral capacities lead to integral solutions in both programs. This observation establishes an analogue of Theorem 1.5, Part I. That is, an analogue of the edge form of Menger's theorem is valid for regular matroids. This has previously been shown by Minty in [24].

It seems likely that the relationship between (3.12) and (3.13) has implications for what is called the "decomposition principle" in linear programming. We shall not pursue this point here.

The only other problem from Part I that we want to examine in the context of Part II is the length—width inequality. (The generalized minimum path problem is the frame—dual of the generalized maximum flow problem and thus presents nothing new. Part III will be devoted to a

very general combinatorial analogue of the maximum capacity path problem.) Let $A = (a_{ij})$ be a real m by n matrix with row space $\mathcal{R}$, and suppose ℓ_j, w_j are given nonnegative numbers for $j = 2, \ldots, n$. Consider the collection $\mathcal{P} = \{P_1, \ldots, P_r\}$ of all elementary vectors of $\mathcal{R}^*$ that have first coordinate 1, and the collection $\mathcal{K} = \{K_1, \ldots, K_s\}$ of all elementary vectors of $\mathcal{R}$ that have first coordinate 1. Let

$$(3.15) \qquad \lambda = \min_{1 \leq i \leq r} \sum_{j=2}^{n} |p_{ij} \ell_j|,$$

$$(3.16) \qquad \omega = \min_{1 \leq k \leq s} \sum_{j=2}^{n} |k_{hj} w_j|,$$

where

$$P_i = (1, p_{i2}, \ldots, p_{in}), \ i = 1, 2, \ldots, r,$$

$$K_h = (1, k_{h2}, \ldots, k_{hn}), \ h = 1, 2, \ldots, s.$$

We call λ the e_1-<u>length</u> of A, and call ω the e_1-<u>width</u> of A.

<u>Theorem 3.4</u>. <u>Let</u> $A = (a_{ij})$ <u>be an</u> m <u>by</u> n <u>real matrix having</u> e_1-<u>length</u> λ <u>relative to</u> $\ell_j \geq 0$, $j = 2, \ldots, n$, <u>and</u> e_1-<u>width</u> ω <u>relative to</u> $w_j \geq 0$, $j = 2, \ldots, n$. <u>Then</u>

$$(3.17) \qquad \lambda\omega \leq \sum_{j=2}^{n} \ell_j w_j .$$

<u>Proof</u>. By Theorem 3.1, Part II (the generalized max-flow min-cut theorem) and the equivalence of (3.12) and (3.13), it follows from (3.16) that there exists a nonnegative r-vector $Y = (y_1, y_2, \ldots, y_r)$ such that

$$(3.17) \qquad \sum_{i=1}^{r} |y_i p_{ij}| \le w_j, \; j = 2, \ldots, n,$$

$$\sum_{i=1}^{r} y_i = \omega.$$

Thus we have

$$\lambda\omega = \lambda \sum_{i=1}^{r} y_i \le \sum_{i=1}^{r} \sum_{j=2}^{n} |p_{ij} \ell_j y_i| \le \sum_{j=2}^{n} w_j \ell_j$$

by (3.18), (3.15), and (3.17), respectively.

Again if A is totally unimodular, then each $P \in \mathscr{P}$ and $K \in \mathscr{K}$ is primitive; taking $\ell_j = w_j = 1$ gives a direct generalization of the Moore—Shannon theorem for graphs to totally unimodular matrices. In matroidal terms:

<u>Corollary 3.5</u>. <u>Let</u> $(E, \mathscr{C})$ <u>be a regular matroid on</u> n <u>edges</u>. <u>Let</u> $\lambda(e) + 1$ <u>be the least number of edges in any circuit containing edge</u> e, $\omega(e) + 1$ <u>the least number of edges in any cocircuit (circuit of the dual matroid) containing</u> e. <u>Then</u> $\lambda(e)\omega(e) \le n - 1$.

PART III - BLOCKING SYSTEMS

In Part I we described the maximum capacity path problem for a two-terminal network, gave a good algorithm for solving it, and presented a min-max theorem concerning paths and cuts for the problem. Similarly, Gross [17] has described a good algorithm and a min-max theorem for the "bottleneck assignment problem": Given a square array of real numbers, find a circling of entries with exactly one circle in each row and in each column so as to maximize the value of the smallest circled entry. For an interpretation, think of rows of the array as corresponding to men, columns to jobs on a serial assembly line, with the entry in row i and column j being the rate at which man i can process items if he is assigned to job j. The theorem established in [17] for this problem is the following: Let $I = \{1, 2, \ldots, n\}$, let $\mathcal{P}$ be the set of permutations of I, let $|C|$ denote cardinality of C, and let a_{ij}, $i \in I$, $j \in I$, be real numbers. Then

$$\max_{P \in \mathcal{P}} \min_{i \in I} a_{i,P(i)} = \min_{\substack{A, B \subset I \\ |A|+|B|=n+1}} \max_{\substack{i \in A \\ j \in B}} a_{ij}.$$

The resemblance between these two min-max theorems is more than superficial. They are, in fact, special cases of a general theorem for a combinatorial structure which might be called a blocking system. These systems have arisen

in numerous contexts (see [21, 22, 29], for example), but the particular axiomatization and general min-max theorem presented in [7] and surveyed here, have apparently not been noted before.

1. AXIOMS AND EXAMPLES

Let E be a finite set, and let $\mathcal{P}$ and $\mathcal{K}$ be two families of subsets of E. We call (E, $\mathcal{P}$, $\mathcal{K}$) a <u>blocking system</u> (on E) if the following two axioms are satisfied:

(1.1) For any partition of E into two sets E_1 and E_0 ($E_0 \cap E_1 = \emptyset$ and $E_0 \cup E_1 = E$), there is either a member of $\mathcal{P}$ contained in E_1 or a member of $\mathcal{K}$ contained in E_0, but not both.

(1.2) No member of $\mathcal{P}$ contains another member of $\mathcal{P}$; no member of $\mathcal{K}$ contains another member of $\mathcal{K}$.

The first axiom (1.1) can be phrased in terms of painting elements of E with two colors: For any blue-red painting of E, there is either a blue P in $\mathcal{P}$ or a red K in $\mathcal{K}$, but not both. The second axiom (1.2) is more a convenience than a necessity for our purposes, as will be clearer later on.

Observe that if (E, $\mathcal{P}$, $\mathcal{K}$) is a blocking system, then for each $P \in \mathcal{P}$ and $K \in \mathcal{K}$, we have $P \cap K \neq \emptyset$, by virtue of the last phrase in (1.1). In other words, each member of $\mathcal{K}$ blocks all members of $\mathcal{P}$, and vice-versa. Note also that the axioms (1.1) and (1.2) are self-dual: Interchanging the roles of $\mathcal{P}$ and $\mathcal{K}$ alters neither (1.1) nor (1.2).

If $\mathcal{P}$ is empty, then $\mathcal{K} = \{\emptyset\}$ satisfies (1.1) and (1.2).

Examples of blocking systems abound. Some reasonably interesting ones will be described. But first we state and prove a theorem that indicates the great profusion of blocking systems. Its proof provides another characterization of blocking systems.

Following [7], we shall call a family $\mathcal{J}$ of subsets of E a clutter on E if no member of $\mathcal{J}$ contains another member of $\mathcal{J}$.

Theorem 1.1. *Let E be a finite set and let $\mathcal{P}$ be a clutter on E. Then there exists a unique clutter $\mathcal{K}$ on E such that (E, $\mathcal{P}$, $\mathcal{K}$) is a blocking system.*

Proof. Let $K \in \mathcal{K}$ if and only if $K \cap P \neq \emptyset$ for all $P \in \mathcal{P}$ and K is minimal with respect to this property. To verify that (E, $\mathcal{P}$, $\mathcal{K}$) is a blocking system, it suffices to check (1.1). Thus consider a blue-red painting of E. Suppose there is no blue $P \in \mathcal{P}$. Let R be the set of all red members of E that belong to some $P \in \mathcal{P}$. Since there is no blue $P \in \mathcal{P}$, we have $R \cap P \neq \emptyset$ for every $P \in \mathcal{P}$. Hence there is a $K \in \mathcal{K}$ such that $K \subset R$, i.e., there is a red $K \in \mathcal{K}$. If there were both a blue $P \in \mathcal{P}$ and a red $K \in \mathcal{K}$, then $P \cap K = \emptyset$, contradicting the definition of $\mathcal{K}$. Thus (1.1) holds and (E, $\mathcal{P}$, $\mathcal{K}$) is a blocking system.

To establish uniqueness, let (E, $\mathcal{P}$, $\mathcal{K}$) and (E, $\mathcal{P}$, $\mathcal{K}'$) be blocking systems on E with $\mathcal{K} \neq \mathcal{K}'$. Interchanging the roles of $\mathcal{K}$ and $\mathcal{K}'$ if necessary, we may suppose $K \in \mathcal{K} - \mathcal{K}'$.

Consider the partition $E - K$, K of E. By (1.1) applied to $(E, \mathcal{P}, \mathcal{K})$, no subset of $E - K$ is a member of $\mathcal{P}$. Hence by (1.1) applied to $(E, \mathcal{P}, \mathcal{K}')$, there is a $K' \in \mathcal{K}'$ with $K' \subset K$. Now consider the partition $E - K'$, K' of E. By (1.2), no subset of K' is a member of $\mathcal{K}$. Hence by (1.1) applied to $(E, \mathcal{P}, \mathcal{K})$, there is a $P' \in \mathcal{P}$ with $P' \subset E - K'$. But then P' and K' violate (1.1) for the blocking system $(E, \mathcal{P}, \mathcal{K}')$ and the partition $E - K'$, K' of E. This contradiction proves Theorem 1.1.

Thus if $\mathcal{P}$ is an arbitrary clutter on E, the family $\mathcal{K} = \mathcal{P}^*$ of all "minimal blockers" of $\mathcal{P}$ is the unique family of Theorem 1.1, and $\mathcal{K}^* = \mathcal{P}^{**} = \mathcal{P}$.

The primary role of (1.2) is to obtain uniqueness in Theorem 1.1. Uniqueness could be achieved in other ways. For instance, instead of normalizing to clutters $\mathcal{P}$ and $\mathcal{K}$ in Theorem 1.1, we could normalize to the families $\mathcal{P}^+$ and $\mathcal{K}^+$ of all supersets of members of $\mathcal{P}$ and $\mathcal{K}$, respectively.

Some examples of blocking systems follow.

Example 1. Let E be the set of edges of a graph G, $\mathcal{P}$ the family of elementary paths joining two vertices of G, and $\mathcal{K}$ the family of elementary cuts separating the two vertices.

Example 2. Let E be the set of cells in an n by n array; let $\mathcal{P}$ be the family of subsets $P \subset E$ having the property that there is just one cell of P in each row and column of the array; let $\mathcal{K}$ be the family of subsets $K \subset E$

such that K is a p by q subarray with $p + q = n + 1$. (That $(E, \mathcal{P}, \mathcal{K})$ is a blocking system follows from a well-known theorem of König [20] which asserts that in an n by n (0, 1)-matrix, the maximum number of 1's, no two of which lie in the same row or column, is equal to the minimum number of rows and columns that contain all the 1's of the array.) More generally, let $\mathcal{P}$ be the family of subsets $P \subset E$ such that $|P| = t$ and P has at most one cell in each row and column. Then $\mathcal{K}$ is the family of subsets $K \subset E$ such that K is a p by q subarray with $p + q = 2n - t + 1$.

Example 3. Let $E = \{1, 2, \ldots, 2k-1\}$, let $\mathcal{P}$ be the family of all k-element subsets of E, and let $\mathcal{K} = \mathcal{P}$. (In multi-person game theory, this example is known as the "straight majority game.")

Example 4. Let E be the set of edges of a graph G, let $\mathcal{P}$ be the family of maximal trees of G, and let $\mathcal{K}$ be the set of all elementary cuts (cocircuits) of G. (A tree of G is a subgraph of G that contains no circuit; a maximal tree is a tree of G that is maximal with respect to this property.)

Example 5. Let E be the set of edges of a graph G, let $\mathcal{P}$ be the family of circuits in G, and let $\mathcal{K}$ be the set of cotrees (complements in E of trees) of G.

Example 6. Let E' be the set of edges of a matroid $(E', \mathcal{C})$, let $E = C' - \{e\}$ for some $e \in E'$, and let $\mathcal{P}$ be the family of subsets P of E such that $\{e\} \cup P \in \mathcal{C}$.

Then $\mathcal{K}$ is the family of subsets K of E such that $\{e\} \cup K \in \mathcal{C}^*$. Here $(E', \mathcal{C}^*)$ is the matroid dual to $(E', \mathcal{C})$.

Example 7. Let E be the set of vertices of a graph G, and let $\mathcal{P}$ be the family of pairs of adjacent vertices of G (two vertices are adjacent if they are joined by an edge.) Then $\mathcal{K}$ is the family of subsets of vertices K such that K covers all edges of G, and is minimal with respect to this property. (In other words, $\mathcal{K}$ is the family of all "minimal blockers" of $\mathcal{P}$.)

It is frequently difficult, as illustrated by Example 7, to find a useful description of the dual clutter $\mathcal{K}$ of a simply described clutter $\mathcal{P}$.

One of the most important problems concerning blocking systems, a problem that arises time and again in applications, is the minimum covering or blocking problem: Given a simple description of $\mathcal{P}$, find a good algorithm that constructs $K \in \mathcal{K}$ such that $|K|$ is a minimum. For example, we might be given $\mathcal{P}$ explicitly, say in the form of an incidence matrix $A = (a(P, e))$, where $a(P, e) = 1$ or 0 according as $e \in P$ or $e \notin P$. The minimum blocking problem then is equivalent to solving the following linear program in integers $x(e) = 0$ or 1:

$$\sum_{e \in E} a(P, e)x(e) \geq 1, \quad \text{all } P \in \mathcal{P}, \tag{1.3}$$

$$\text{minimize } \sum_{e \in E} x(e).$$

Various methods have been proposed for such problems, but no good algorithms are known. Indeed, most of the methods that have been proposed can be shown to be bad: the amount of computational effort increases exponentially with the size of the problem.

There is a good algorithm, however, for computing the following lower bound on the minimum in (1.3). Consider the class $\mathcal{A}$ of all (0, 1)-matrices having the same row and column sums as A. For A in $\mathcal{A}$, let $\omega(A)$ denote the minimum in (1.3), and let

$$\tilde{\omega} = \min_{A \in \mathcal{A}} \omega(A). \tag{1.4}$$

The integer $\tilde{\omega}$ has been explicitly evaluated by Fulkerson and Ryser in [14], and a very simple construction for a matrix $\tilde{A}$ in $\mathcal{A}$ such that $\omega(\tilde{A}) = \tilde{\omega}$ has been given in [15].

2. THE MIN-MAX THEOREM

The analogue of Theorem 3.1, Part I, is valid for all blocking systems, and can be viewed as characterizing blocking systems:

Theorem 2.1. Let $(E, \mathcal{P}, \mathcal{K})$ be a blocking system, and let f be a real-valued function defined on E. Then

$$\max_{P \in \mathcal{P}} \min_{e \in P} f(e) = \min_{K \in \mathcal{K}} \max_{e \in K} f(e). \tag{2.1}$$

Conversely, if $\mathcal{P}$ and $\mathcal{K}$ are clutters on E such that (2.1)

holds for every real-valued f *defined on* E, *then* $(E, \mathcal{P}, \mathcal{K})$ *is a blocking system*.

Proof. The proof that (2.1) holds for a blocking system is entirely analogous to the proof of Theorem 3.1, Part I. In brief: The left-hand side of (2.1) is less than or equal to the right-hand side since $P \cap K$ is nonempty for each $P \in \mathcal{P}$, $K \in \mathcal{K}$. To establish equality, order the elements of E according to decreasing values of f; then paint elements of E blue, one after another, until the blue set first contains an element of $\mathcal{P}$.

(In other words, the threshold method establishes equality in (2.1) and simultaneously evaluates (2.1). It will be a good method for this evaluation in case there is a good method for recognizing whether an arbitrary subset of E contains a member of $\mathcal{P}$ (or a member of $\mathcal{K}$).)

Conversely, let $\mathcal{P}$ and $\mathcal{K}$ be clutters on E and suppose (2.1) holds for every real-valued f defined on E. Let $f(e) = 1$ or 0 according as e is blue or red. Suppose there is no blue $P \in \mathcal{P}$. Then

$$\max_{P \in \mathcal{P}} \min_{e \in P} f(e) = 0 = \min_{K \in \mathcal{K}} \max_{e \in K} f(e).$$

If there were no red $K \in \mathcal{K}$, we would have

$$\min_{K \in \mathcal{K}} \max_{e \in K} f(e) = 1,$$

a contradiction. Hence there is a red $K \in \mathcal{K}$. On the other hand, if there were both a blue $P \in \mathcal{P}$ and a red $K \in \mathcal{K}$, then

$$\max_{P\in\mathcal{P}} \min_{e\in P} f(e) = 1, \quad \min_{K\in\mathcal{K}} \max_{e\in K} f(e) = 0,$$

contradicting (2.1). Hence $(E, \mathcal{P}, \mathcal{K})$ is a blocking system.

3. THE LENGTH-WIDTH INEQUALITY AND MAX-FLOW MIN-CUT EQUALITY

Let $(E, \mathcal{P}, \mathcal{K})$ be a blocking system, and suppose $\ell(e)$, $w(e)$ are two nonnegative numbers associated with element $e \in E$. Define the <u>length</u> of the system to be

$$\lambda = \min_{P\in\mathcal{P}} \sum_{e\in P} \ell(e), \tag{3.1}$$

and the <u>width</u> to be

$$\omega = \min_{K\in\mathcal{K}} \sum_{e\in K} w(e). \tag{3.2}$$

Following Lehman [22], we shall say that the length-width inequality holds for $(E, \mathcal{P}, \mathcal{K})$ if

$$\lambda\omega \leq \sum_{e\in E} \ell(e)w(e) \tag{3.3}$$

is satisfied for every pair of nonnegative functions ℓ, w defined on E.

For instance, if $(E, \mathcal{P}, \mathcal{K})$ is the blocking system of Example 1, we have seen in Part I that the length-width inequality holds. It also holds for Example 6 provided the underlying matroid is regular; this is a corollary of Theorem 3.4, Part II. On the other hand, the length-width inequality fails for the blocking system of Example 3.

For each $P \in \mathcal{P}$ and $e \in E$, define $a(P, e) = 1$ or 0 according as $e \in P$ or $e \notin P$. Now consider the linear program

$$(3.3) \qquad \sum_{P \in \mathcal{P}} y(P)a(P,e) \le w(e), \qquad e \in E,$$

$$y(P) \ge 0, \qquad P \in \mathcal{P},$$

$$\text{maximize} \sum_{P \in \mathcal{P}} y(P).$$

Clearly the maximum in (3.3) is less than or equal to the width of $(E, \mathcal{P}, \mathcal{K})$. If equality holds here for every nonnegative w defined on E, we say, as in [22], that the max-flow min-cut equality holds for $(E, \mathcal{P}, \mathcal{K})$.

Thus, for instance, the max-flow min-cut equality holds for Example 1, for Example 6 if the underlying matroid is regular, and fails for Example 3, just as for the length-width inequality.. This behavior is not accidental. One of the main results of [22] is that the max-flow min-cut equality holds for a blocking system

if and only if the length-width inequality holds. Consequently, if the max-flow min-cut equality holds for $(E, \mathcal{P}, \mathcal{K})$, it also holds for $(E, \mathcal{K}, \mathcal{P})$, since the roles of $\mathcal{P}$ and $\mathcal{K}$ are symmetric in the length-width inequality.

In any event, the problem of evaluating the width of a blocking system for a given nonnegative function w is a generalization of the minimum blocking problem mentioned earlier. It would be interesting to discover other significant classes of blocking systems for which the length-width inequality, and hence the max-flow min-cut equality, holds.

BIBLIOGRAPHY

1. Berge, C. and A. Ghouila-Houri, Programmes, Jeux et Réseaux de Transport, Dunod, Paris, 1962.

2. Camion, P., "Application d'une généralization du lemme de Minty à un problème d'infinium de function convexe," Cahiers Centre Études Recherche Opér., 7, 1965, 230-247.

3. Dantzig, G. B., Linear Programming and Extensions, Princeton University Press, Princeton, New Jersey, 1963.

4. Dantzig, G. B.,and D. R. Fulkerson, "On the Max-Flow Min-Cut Theorem of Networks, Linear Inequalities and Related Systems, Ann. of Math. Study 38, Princeton University Press, Princeton, New Jersey, 1956, 215-221.

5. Duffin, R. J., "The Extremal Length of a Network," J. Math. Anal. and Appl., 5, 1962, 200-215.

6. Edmonds, J., "Paths, Trees, and Flowers," Canadian J. Math., 17, 1965, 449-467.

7. Edmonds, J. and D. R. Fulkerson, Bottleneck Extrema (to appear).

8. Ford, L. R., Jr. and D. R. Fulkerson, Flows in Networks, Princeton University Press, Princeton, New Jersey, 1962.

9. ————————, "Maximal Flow Through a Network," Canadian J. Math., 1956, 399-404.

10. ————————, "A Simple Algorithm for Maximal Network Flows and an Application to the Hitchcock Problem," Canadian J. Math., 9, 1957, 210-218.

11. ————————, "Constructing Maximal Dynamic Flows from Static Flows," Op. Res., 6, 1958, 419-433.

12. Fulkerson, D. R., "An Out-of-Kilter Method for Minimal Cost Flow Problems," J. Soc. Indust. Appl. Math., 9, 1961, 18-27.

13. ————————, "Flow Networks and Combinatorial Operations Research," Amer. Math. Monthly, 73, 1966, 115-138.

14. Fulkerson, D. R.and H. J. Ryser, "Widths and Heights of (0, 1)-Matrices," Canadian J. Math., 13, 1961, 239-255.

15. ————————, "Multiplicities and Minimal Widths for (0, 1)-Matrices," Canadian J. Math., 14, 498-508.

16. Gale, D., H. W. Kuhn, and A. W. Tucker, "Linear Programming and the Theory of Games," Activity Analysis of Production and Allocation, John Wiley and Sons, New York, 1951, 317-329.

17. Gross, O., The Bottleneck Assignment Problem, The RAND Corporation, P-1630, 1959.

18. Hoffman, A. J., "Some Recent Applications of the Theory of Linear Inequalities to Extremal Combinatorial Analysis," Proc. Symposia Applied Math., 10, 1960.

19. Hu, T. C., "The Maximum Capacity Route Problem," Op. Res., 9, 1961, 898-900.

20. Konig, D., Theorie der Endlichen und Unendlichen Graphen, Chelsea, New York, 1950.

21. Lawler, E., "Covering Problems: Duality Relations and a New Method of Solution," SIAM Journal, 14, 1966, 1115-1133.

22. Lehman, A., On the Width Length Inequality (mimeographed).

23. Minty, G. J., "Monotone Networks," Proc. Roy. Soc., A, 257, 1960, 194-212.

24. ——————, "On the Axiomatic Foundations of the Theories of Directed Linear Graphs, Electrical Networks, and Network-Programming," J. Math. Mech. 15, 1966, 485-520.

25. Moore, E. F. and C. E. Shannon, "Reliable Circuits Using Less Reliable Relays," J. Franklin Inst. 262, 1956, 191-208.

26. Pollack, M., "The Maximum Capacity Route Through a Network," Op. Res. 8, 1960, 733-736.

27. Robacker, J. T., Min-Max Theorems on Shortest Chains and Disjunct Cuts of a Network, The RAND Corporation, RM-1660-PR, 1956.

28. Rockafellar, R. T., "The Elementary Vectors of a Subspace of R^N," to appear in the Proceedings of the Chapel Hill Symposium on Combinatorial Mathematics and Its Applications.

29. Shapley, L. S., "Simple Games: An Outline of the Descriptive Theory," Behavorial Sciences, 7, 1962, 59-66.

30. Tutte, W. T., "A Class of Abelian Groups," Canadian J. Math., 8, 1956, 13-28.

31. ——————, "A Homotopy Theorem for Matroids I, II," Trans. Amer. Math. Soc., 88, 1958, 144-174.

32. ——————, "Matroids and Graphs," Trans. Amer. Math. Soc., 90, 1959, 527-552.

33. ——————, "Lectures on Matroids," J. Res. Nat. Bur. Std., B, 69, 1965, 1-47.

34. ——————, Introduction to the Theory of Matroids, The RAND Corporation, R-448-PR, 1966.

35. Whitney, H., "On the Abstract Properties of Linear Dependence," Amer. J. Math., 57, 1935, 507-553.

DOCUMENT CONTROL DATA

1. ORIGINATING ACTIVITY THE RAND CORPORATION	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP
3. REPORT TITLE NETWORKS, FRAMES, BLOCKING SYSTEMS	
4. AUTHOR(S) (Last name, first name, initial) Fulkerson, D. R.	

5. REPORT DATE May 1967	6a. TOTAL No. OF PAGES 66	6b. No. OF REFS. 35
7. CONTRACT OR GRANT No. F44620-67-C-0045	8. ORIGINATOR'S REPORT No. RM-5368-PR	
9a. AVAILABILITY/LIMITATION NOTICES DDC-1	9b. SPONSORING AGENCY United States Air Force Project RAND	

10. ABSTRACT

A survey of some of the basic problems, theorems, and constructions for flow networks, and their extension to more general combinatorial structures. One generalization is that obtained by replacing the vertex-edge incidence matrix of an oriented network by an arbitrary real matrix. This leads to the notion of a frame of a subspace of Euclidean n-space, a concept very closely allied to that of a real matric matroid. The treatment relates matroid theory and linear programming theory, and thus provides another viewpoint on linear programming, and in particular, on digraphoid-programming. In addition, a very general combinatorial structure called a blocking system is given an axiomatic formulation. These systems have arisen in a variety of contexts, including multiperson game theory and abstract covering problems. It is shown that one of the network theorems surveyed in the study extends to all blocking systems, and indeed characterizes such systems.

11. KEY WORDS

Blocking systems
Matroid theory
Network theory
Linear programming
Digraphoid programming
Mathematics
Game theory